ON BENDING OF AN INFINITE BEAM ON ELASTIC HALF-SPACE

(OB IZGIBE BESKONECHNOI BALKI NA UPRYGOM POLUPROSTRANTSTVE)

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This paper deals with the application of solutions of a three-dimensional contact problem of a strip to the analysis of an infinite beam resting on an elastic foundation. Instead of the commonly used Hertz theory, this paper assumes that between the deflection $w(x, y, 0)$ and the load $p(x, y)$ the following relationship holds

$$w(x, y, 0) = \frac{1 - \nu_0^2}{\pi E_0} \iint\limits_{(S)} \frac{p(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} + kp(x, y)$$

where ν_0 and E_0 are elastic constants of the material; (S) is a part of a half-space on which the load $p(x, y)$ is acting, k is some constant depending on the structure of the surface of the lastic support.

This generalization of the Hertz theory was proposed by Shtaerman [1]. It represents a combination of the Hertz theory and the "bedding coefficient" hypothesis, and it includes them as particular cases.

The above problem is solved by assuming that the base of the beam before deformation is plane and the bending takes place in the longitudinal direction only.

1. The differential equation of the bending of the beam has the following form:

$$EIw^{\mathrm{IV}}(y) = q(y) - r(y) \tag{1.1}$$

where E is the modulus of elasticity of the beam, I the moment of inertia of the beam cross-section, $w(y)$ the deflection of the axis of the beam, $q(y)$ the load, $r(y)$ the reaction of the support per unit length.

If the beam has contact with the elastic support along the strip (S), $|x| \leqslant a$, $-\infty < y < \infty$, then equation (1) takes the following form:

$$w(x, y, 0) = \frac{1-\nu_0^2}{\pi E_0} \int_{-a}^{a} d\xi \int_{-\infty}^{\infty} \frac{p(\xi, \eta)\, d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}} + kp(x,y) \tag{1.2}$$

Note that $w(x, y, 0) = w(y)$ in the region (S), moreover

$$r(y) = \int_{-a}^{a} p(x, y)\, dx \tag{1.3}$$

Let us consider first the case when $q(y) = a(\lambda) \cos \lambda y$, where $\lambda > 0$ is an arbitrary parameter and $a(\lambda)$ is an arbitrary function of λ. In this case the system of equations (1.1) and (1.2) can be satisfied by putting

$$w(y) = b(\lambda) \cos \lambda y, \qquad r(y) = c(\lambda) \cos \lambda y \tag{1.4}$$

where $b(\lambda)$ and $c(\lambda)$ are some functions of λ.

It is easy to verify that equation (1.1) is satisfied if

$$EI\lambda^4 b(\lambda) = a(\lambda) - c(\lambda) \tag{1.5}$$

Taking into account that $w(x, y, 0) = b(\lambda) \cos \lambda y$ for $|x| < a$, equation (1.2) is satisfied if we put

$$p(x, y) = \varphi(\lambda, x) \cos \lambda y \tag{1.6}$$

where $\phi(\lambda, x)$ is a solution of the following equation

$$b(\lambda) = \frac{2(1-\nu_0^2)}{\pi E_0} \int_{-a}^{a} \varphi(\lambda, t) K_0(\lambda |x-t|)\, dt + k\varphi(\lambda, x) \tag{1.7}$$

In the last formula $K_0(t)$ is a well-known Bessel function. After finding $\phi(\lambda, x)$ and noticing that

$$\int_{-a}^{a} \varphi(\lambda, x)\, dx = c(\lambda) \tag{1.8}$$

we obtain a condition which must be fulfilled by the constants $b(\lambda)$ and $c(\lambda)$ for (1.4) to be satisfied.

2. Since the function $K_0(t)$ satisfies

$$y'' + t^{-1}y' - y = 0$$

it is possible to prove that the function

$$\Phi(\lambda, x, z) = \frac{2(1-\nu_0^2)}{\pi E_0} \int_{-a}^{a} \varphi(\lambda, t) K_0[\lambda \sqrt{(x-t)^2+z^2}]\, dt \tag{2.1}$$

satisfies equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} - \lambda^2 \Phi = 0 \tag{2.2}$$

at all points of the xz-plane except for the points of a segment $[-a, a]$ of the x-axis, and tends to zero at infinity.

Since the function $K_0(t)$ has only one singularity (logarithmic) at $t = 0$, it is possible to obtain the following formulas, analogous to the well-known formulas in potential theory:

$$\begin{aligned}\frac{\partial\Phi(\lambda, x, +0)}{\partial z} &= \frac{2(1-\nu_0^2)}{\pi E_0}\varphi(\lambda, x) \\ \frac{\partial\Phi(\lambda, x, -0)}{\partial z} &= -\frac{2(1-\nu_0^2)}{\pi E_0}\varphi(\lambda, x)\end{aligned} \qquad (|x|<a) \tag{2.3}$$

$$\frac{\partial\Phi(\lambda, x, 0)}{\partial z} = 0 \qquad (|x|>a) \tag{2.4}$$

It follows from formulas (2.3) and (1.7) that for the points of the segment $|x| < a$ of the x-axis the following is true:

$$\Phi(\lambda, x, 0) + k_1\frac{\partial\Phi(\lambda, x, 0)}{\partial z} = b(\lambda) \qquad \left(k_1 = \frac{\pi E_0 k}{2(1-\nu_0^2)}\right) \tag{2.5}$$

Thus, the problem is reduced to finding a solution of the equation (2.2) which satisfies boundary conditions (2.5) and becomes zero at infinity.

In the author's paper [2] a solution was found for the case when $k_1 = 0$. Therefore, let $\Phi_0(\lambda, x, z)$ be a solution which corresponds to this particular case. For $|x| < a$, we then have

$$\Phi_0(\lambda, x, 0) = b(\lambda), \qquad \frac{\partial\Phi_0(\lambda, x, +0)}{\partial z} = \frac{2b(\lambda)}{\pi\sqrt{a^2-x^2}}\sum_{\nu=0}^{\infty}\delta_{2\nu}\cos 2\nu\cos^{-1}\frac{x}{a} \tag{2.6}$$

where

$$\delta_{2\nu} = (-1)^\nu\sum_{i=0}^{\infty}\frac{A_0^{(2i)}A_{2\nu}^{(2i)}\mathrm{Fek}'_{2i}(0, -{}^1/_4 a^2\lambda^2)}{\mathrm{Fek}_{2i}(0, -{}^1/_4 a^2\lambda^2)} \tag{2.7}$$

The numbers $A_{2\nu}^{(2i)}$ are Fourier coefficients of the Mathieu functions $ce_{2i}(x, -1/4\ a^2\lambda^2)$, and $\mathrm{Fek}_{2i}(x, -q)$ are well known Mathieu functions [3]. To calculate δ_{2i} we can use Tables [4]. Next, by applying Green's formula we get

$$\iint\limits_{(D)}(v\nabla u - u\nabla v)\,dx\,dy = \int\limits_{[L]}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds$$

where (D) represents the interior of a circle with an arbitrary radius R and without the points of the segment $[-a, a]$ of the x-axis. Putting $v = \Phi(\lambda, x, z)$ and $u = \Phi_0(\lambda, x, z)$, and taking into consideration the fact that these functions satisfy (2.2), as $R \to \infty$, we get

$$\int_{-a}^{a}\Phi(\lambda, x, 0)\frac{\partial\Phi_0(\lambda, x, +0)}{\partial z}dx - \int_{-a}^{a}\Phi_0(\lambda, x, 0)\frac{\partial\Phi(\lambda, x, +0)}{\partial z}dx = 0 \tag{2.8}$$

and because of (2.5) we get

$$b(\lambda)\int_{-a}^{a}\Phi_{0z}'\,dx - k_1\int_{-a}^{a}\Phi_z'\Phi_{0z}'dx - b(\lambda)\int_{-a}^{a}\Phi_z'dx = 0 \tag{2.9}$$

It follows from (2.9) that

$$b(\lambda)\int_{-a}^{a}\Phi_{0z}'\,dx - \frac{k_1}{2a}\int_{-a}\Phi_z'\,dx\int_{-a}\Phi_{0z}'\,dx +$$
$$+k_1\int_{-a}^{a}\Phi_{0z}'\left(\frac{1}{2a}\int_{-a}^{a}\Phi_z'\,dx - \Phi_z'\right)dx - b(\lambda)\int_{-a}^{a}\Phi_z'dx = 0 \tag{2.10}$$

From (2.10) we obtain

$$\int_{-a}^{a}\Phi_z'\,dx = \frac{b(\lambda)J}{b(\lambda)+(k_1/2a)J} + \varepsilon \qquad \left(J = \int_{-a}^{a}\Phi_{0z}'\,dx\right) \tag{2.11}$$

where

$$\varepsilon = \frac{k_1}{b(\lambda)+(k_1/2a)J}\int_{-a}^{a}\Phi_{0z}'\left(\frac{1}{2a}\int_{-a}^{a}\Phi_z'dx - \Phi_z\right)(k_1/2a)\,Jdx \tag{2.12}$$

It is easy to verify that $\Phi_z' \to$ const as $k_1 \to \infty$, and consequently $\epsilon \to 0$ as $k \to \infty$. Moreover, $\epsilon \to 0$ as $k_1 \to 0$. Since $\Phi'_{z0} \to$ const for $\lambda \to \infty$, then $\epsilon \to 0$ for $\lambda \to \infty$. Thus, for sufficiently large and sufficiently small values of k_1 we can neglect ϵ in (2.11). It should be mentioned, however, that by dropping ϵ in this formula we commit certain errors in evaluating the magnitude of k.

An exact solution of the problem for all values of k is connected with a solution of the above boundary value problem, and it may be an object for further investigations.

Note that

$$\int_{-a}^{a}\Phi_{0z}'\,dx = \frac{2b(\lambda)}{\pi}\sum_{\nu=0}^{\infty}\delta_{2\nu}\int_{-a}^{a}\frac{\cos 2\nu\cos^{-1}\frac{x}{a}}{\sqrt{a^2-x^2}}\,dx = 2b(\lambda)\,\delta_1 \tag{2.13}$$

From (1.8), (2.3) and (2.11) we get

$$c(\lambda) \approx \frac{\pi E_0 a\delta_0}{(1-\nu_0^2)(a+k_1\delta_0)} \tag{2.14}$$

and it follows from here that

$$b(\lambda) \approx \frac{(1-\nu_0^2)[R(a\lambda)+\lambda k_1]}{\pi E_0 a\lambda}\,c(\lambda) \qquad \left(R(a\lambda) = \frac{a\lambda}{\delta_0}\right) \tag{2.15}$$

Some values of the function $R(t)$ are shown in the Table below:

TABLE 1.

t	$R(t)$	t	$R(t)$	t	$R(t)$	t	$R(t)$
0	0	0.5	1.600	1.0	2.100	2.0	2.514
0.1	0.624	0.6	1.733	1.2	2.213	4.0	2.793
0.2	0.977	0.7	1.846	1.4	2.316	6.0	2.900
0.3	1.235	0.8	1.944	1.6	2.395	∞	π
0.4	1.436	0.9	2.027	1.8	2.459		

Eliminating $c(\lambda)$ from (1.5) and (2.15) we get

$$b(\lambda) = \frac{(1-\nu_0^2)[R(a\lambda) + \lambda k_1]}{\pi E_0 a\lambda + EI(1-\nu_0^2)[R(a\lambda) + \lambda k_1]\lambda^4}\, a(\lambda) \qquad (2.16)$$

3. It follows from (2.17) that to the load $q(y) = a(\lambda)\cos\lambda y$ there corresponds the following deflection

$$w(y) = \frac{(1-\nu_0^2)[R(a\lambda) + \lambda k_1]\, a(\lambda)}{\pi E_0 a\lambda + EI(1-\nu_0^2)[R(a\lambda) + \lambda k_1]\lambda^4}\cos\lambda y \qquad (3.1)$$

Representing an arbitrary load in the form of a Fourier integral we can find the deflection of a beam from the following formula

$$w(y) = \frac{(1-\nu_0^2)a^3}{\pi}\int_0^\infty \frac{[aR(u) + k_1 u]\, du}{a^5\pi E_0 u + EI(1-\nu_0^2)[aR(u) + k_1 u]u^4}\int_{-\infty}^{+\infty} q(t)\cos\frac{u}{a}(y-t)\, dt \qquad (3.2)$$

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